- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$X(t) \overset{\text{CTFT}}{\longleftrightarrow} 2\pi x(-\omega).$$
- This is known as the duality property of the Fourier transform.
- This property follows from the high degree of symmetry in the forward and inverse Fourier transform equations, which are respectively given by
$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta) e^{-j\theta\lambda} d\theta \quad \text{and} \quad x(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta) e^{j\theta\lambda} d\theta.$$
- That is, the forward and inverse Fourier transform equations are identical except for a *factor of* 2π and *different sign* in the parameter for the exponential function.
- Although the relationship $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$ only directly provides us with the Fourier transform of $x(t)$, the duality property allows us to indirectly infer the Fourier transform of $X(t)$. Consequently, the duality property can be used to effectively *double* the number of Fourier transform pairs that we know.

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1 * x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega) X_2(\omega).$$

- This is known as the convolution (or time-domain convolution) property of the Fourier transform.
- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.
- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t)x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{2\pi} X_1 * X_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)X_2(\omega-\theta)\,d\theta.$$

- This is known as the multiplication (or frequency-domain convolution) property of the Fourier transform.
- In other words, multiplication in the time domain becomes convolution in the frequency domain (up to a scale factor of 2π).
- Do not forget the factor of $\frac{1}{2\pi}$ in the above formula!
- This property of the Fourier transform is often tedious to apply (in the forward direction) as it turns a multiplication into a convolution.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\frac{dx(t)}{dt} \overset{\text{CTFT}}{\longleftrightarrow} j\omega X(\omega).$$

- This is known as the differentiation property of the Fourier transform.
- Differentiation in the time domain becomes multiplication by $j\omega$ in the frequency domain.
- Of course, by repeated application of the above property, we have that $\left(\frac{d}{dt}\right)^n x(t) \overset{\text{CTFT}}{\longleftrightarrow} (j\omega)^n X(\omega)$.
- The above suggests that the Fourier transform might be a useful tool when working with differential (or integro-differential) equations.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$tx(t) \overset{\text{CTFT}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega).$$

- This is known as the frequency-domain differentiation property of the Fourier transform.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$

- This is known as the integration property of the Fourier transform.
- Whereas differentiation in the time domain corresponds to *multiplication* by $j\omega$ in the frequency domain, integration in the time domain is associated with *division* by $j\omega$ in the frequency domain.
- Since integration in the time domain becomes division by $j\omega$ in the frequency domain, integration can be easier to handle in the frequency domain.
- The above property suggests that the Fourier transform might be a useful tool when working with integral (or integro-differential) equations.

- Recall that the energy of a signal x is given by $\int_{-\infty}^{\infty} |x(t)|^2\, dt$.
- If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

(i.e., the energy of x and energy of X are equal up to a factor of 2π).

- This relationship is known as Parseval's relation.
- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform *preserves energy* (up to a scale factor).

- For a signal x with Fourier transform X, the following assertions hold:

$$x \text{ is even} \Leftrightarrow X \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow X \text{ is odd}.$$

- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

- A signal x is *real* if and only if its Fourier transform X satisfies

$$X(\omega) = X^*(-\omega) \quad \text{for all } \omega$$

(i.e., X has *conjugate symmetry*).

- Thus, for a real-valued signal, the portion of the graph of a Fourier transform for negative values of frequency ω is *redundant*, as it is completely determined by symmetry.
- From properties of complex numbers, one can show that $X(\omega) = X^*(-\omega)$ is equivalent to

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega)$$

(i.e., $|X(\omega)|$ is *even* and $\arg X(\omega)$ is *odd*).

- Note that x being real does *not* necessarily imply that X is real.

- The Fourier transform can be generalized to also handle periodic signals.
- Consider a periodic signal x with period T and frequency $\omega_0 = \frac{2\pi}{T}$.
- Define the signal x_T as

$$x_T(t) = \begin{cases} x(t) & \text{for } -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases}$$

(i.e., $x_T(t)$ is equal to $x(t)$ over a single period and zero elsewhere).

- Let a denote the Fourier series coefficient sequence of x.
- Let X and X_T denote the Fourier transforms of x and x_T, respectively.
- The following relationships can be shown to hold:

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0),$$

$$a_k = \frac{1}{T} X_T(k\omega_0), \quad \text{and} \quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0).$$

- The Fourier series coefficient sequence a_k is produced by sampling X_T at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$.
- The Fourier transform of a periodic signal can only be nonzero at integer multiples of the fundamental frequency.

Section 5.4

Fourier Transform and Frequency Spectra of Signals

- Like Fourier series, the Fourier transform also provides us with a frequency-domain perspective on signals.
- That is, instead of viewing a signal as having information distributed with respect to *time* (i.e., a function whose domain is time), we view a signal as having information distributed with respect to *frequency* (i.e., a function whose domain is frequency.(
- The Fourier transform of a signal x provides a means to *quantify* how much information x has at different frequencies.
- The distribution of information in a signal over different frequencies is referred to as the *frequency spectrum* of the signal.

- To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with $X(\omega)$ expressed in *polar form* as follows:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\,e^{j[\omega t+\arg X(\omega)]}\,d\omega.$$

- In effect, the quantity $|X(\omega)|$ is a *weight* that determines how much the complex sinusoid at frequency ω contributes to the integration result x.
- Perhaps, this can be more easily seen if we express the above integral as the *limit of a sum*, derived from an approximation of the integral using the area of rectangles, as shown on the next slide. [Recall that $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\Delta x \to 0}\sum_{k=-\infty}^{\infty} \Delta x f(k\Delta x)$.]

- Expressing the integral (from the previous slide) as the *limit of a sum*, we obtain

$$x(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega \left| X(\omega') \right| e^{j[\omega' t + \arg X(\omega')]},$$

where $\omega' = k\Delta\omega$.

- In the above equation, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $\omega' = k\Delta\omega$ that has had its *amplitude scaled* by a factor of $|X(\omega')|$ and has been *time shifted* by an amount that depends on $\arg X(\omega')$.
- For a given $\omega' = k\Delta\omega$ (which is associated with the kth term in the summation), the larger $|X(\omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega' t}$ will be, and therefore the larger the contribution the kth term will make to the overall summation.
- In this way, we can use $|X(\omega')|$ as a *measure* of how much information a signal x has at the frequency ω'.